TABLE VI. Pressure derivatives of polycrystalline elastic moduli at different thermodynamic boundary conditions (at 298°K).

Pressure derivatives	dB/dp	dG/dp	dL/dp
$(\partial M^s/\partial p)_T$	4.19[a] 3.70[b]	1.79[a]	6.57[a]
$(\partial M^T/\partial p)_T$	4.23 4.10[c]	1.79	6.62
$(\partial M^s/\partial p)_s$	4.16	1.73	6.52

[a] These are taken from Table V.

[b] This is calculated from the Dugdale–MacDonald relation, i.e., $(\partial B^s/\partial p)_T=2\gamma_G+1$, where γ_G is the Grüneisen parameter.

[c] This was obtained from the Murnaghan equation of state by a curve-fitting procedure using experimental data on compression.

Table VI, two other values of pressure derivative of the bulk modulus have been listed. One is a theoretical value based on the Dugdale–MacDonald relation,[23] and the other is derived from the Murnaghan equation of state[24] by a curve-fitting procedure using experimental data on compression.[9–11] A detailed discussion on these quantities will follow in Sec. 5. It is seen, however, that these values compare reasonably well with the corresponding quantities resulting from the ultrasonic-pressure experiments made on both the single-crystal and polycrystalline materials.

4.2. Variation with Temperature

Recalling Eq. (4.1), the temperature derivative of the porosity-sensitive elastic modulus is

$$dM/dT=(dM_0/dT)(1-\alpha\eta)-M_0\alpha(d\eta/dT). \quad (4.18)$$

Since $(d\eta/dT)$ is zero, the last term drops out. Thus, dividing the resulting part of Eq. (4.18) by Eq. (4.1), we obtain

$$(1/M)(dM/dT)=(1/M_0)(dM_0/dT). \quad (4.19)$$

Equation (4.19) implies that the temperature coefficient of an elastic modulus determined on a porous polycrystalline specimen can be used to estimate the elastic modulus of the nonporous polycrystalline aggregates (as a function of temperature) simply by interpolating the room-temperature modulus of the porous aggregate to that of the nonporous aggregate. The result of such interpolations is given in Table VII. This represents the isotropic elastic parameters of polycrystalline alumina as a function of temperature, which are to be compared with the corresponding single-crystal data.

Figure 3 illustrates the comparison of polycrystalline longitudinal modulus with the corresponding quantity calculated from the single-crystal data using the Voigt–Reuss–Hill (VRH) approximation. The single-crystal data used here are those of Tefft.[7] Figure 4 is a similar comparison for the isotropic shear modulus. It is evident that the comparison is good for the shear modulus throughout the temperature range considered. For the longitudinal modulus, we note that there is a significant discrepancy between our data and the VRH modulus calculated from Tefft's single-crystal data (particularly at low temperatures). At present, it is very difficult to see why the longitudinal modulus at temperatures below 100°K calculated from the single-crystal data decreases with decreasing temperature. Our measurements on polycrystalline specimens indicate exactly the opposite behavior so that the elastic moduli become *stiffer* as temperature decreases.

5. INTERPRETATION AND DISCUSSION

5.1. Analysis of Temperature Dependence of the Isotropic Elastic Moduli

The total temperature dependence of an elastic modulus can be thought of as consisting of two parts: one, a result of an explicit temperature change, and the other an implicit part resulting from a change in volume with temperature. In other words, the elastic modulus

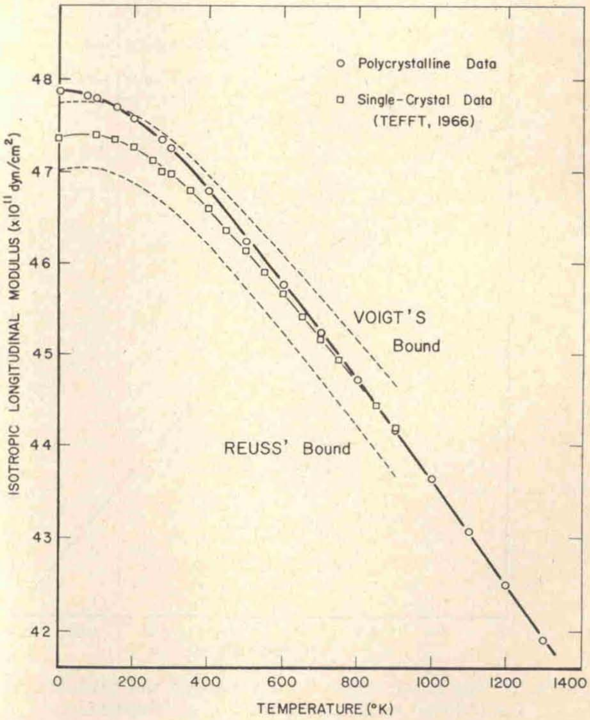

FIG. 3. A comparison between the measured and calculated isotropic longitudinal modulus as a function of temperature.

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[23] J. S. Dugdale and D. K. C. MacDonald, Phys. Rev. **89,** 832 (1953).

[24] F. D. Murnaghan, Proc. Natl. Acad. Sci. **30,** 244 (1944); F. D. Murnaghan, *Non-Linear Problems in Mechanics of Continua, Proceedings of the Symposium on Applied Mathematics* (American Mathematical Society, Providence, R.I., 1949) Vol. 1.

TABLE VII. Thermo-elastic properties of polycrystalline α-Al_2O_3 (at 1 atm).

Temperature (°K)	Density (g/cm³)	Velocities			Elastic moduli			$\theta_{D(\text{elastic})}$ (°K)
		v_l	v_t (km/sec)	v_m	L^s	($\times 10^{11}$ dyn/cm²)	B^s	
4.2	3.9924	10.951	6.448	7.146	47.879	16.599	25.747	1044(±3)
77	3.9920	10.947	6.446	7.144	47.839	16.587	25.727	1043
100	3.9918	10.942	6.444	7.141	47.793	16.576	25.691	1042.6
150	3.9910	10.931	6.437	7.134	47.687	16.537	25.638	1041
200	3.9896	10.919	6.427	7.123	47.566	16.480	25.593	1040
273	3.9863	10.896	6.408	7.102	47.326	16.369	25.512	1036
300	3.986	10.889	6.398	7.092	47.262	16.362	25.507	1035
400	3.978	10.844	6.368	7.059	46.778	16.131	25.270	1029
500	3.969	10.798	6.333	7.021	46.277	15.918	25.053	1023
600	3.960	10.750	6.296	6.981	45.763	15.697	24.833	1016
700	3.949	10.702	6.259	6.941	45.229	15.470	24.602	1010
800	3.939	10.653	6.218	6.896	44.702	15.230	24.396	1002
900	3.928	10.602	6.178	6.853	44.152	14.992	24.162	995
1000	3.918	10.552	6.137	6.809	43.625	14.756	23.950	988
1100	3.907	10.498	6.096	6.764	43.058	14.519	23.700	981
1200	3.897	10.444	6.053	6.718	42.507	14.278	23.470	972
1300	3.886	10.386	6.011	6.667	41.918	14.041	23.197	965

M of Born–von Karman-type solids can be treated as a function of volume V (interatomic separation) and temperature T[25]:

$$M=M(V,T). \tag{5.1}$$

Taking logarithms and differentiating both sides with respect to pressure, and rearranging the results, we find

$$d(\ln M)/dT=-\beta B^T[\partial(\ln M)/\partial p]_T+[\partial(\ln M)/\partial T]_V, \tag{5.2}$$

where B^T is the isothermal bulk modulus. Hence, from our data on both the pressure and temperature dependences of the isotropic elastic moduli, we should be able to separate out the *changes* due to temperature from those due to volume. Rewriting Eq. (5.2) for the explicit term

$$[\partial(\ln M)/\partial T]_V=d(\ln M)/dT+\beta B^T[\partial(\ln M)/\partial p]_T.$$

$$(\text{explicit})=(\text{total})+(\text{implicit}). \tag{5.3}$$

In Table VIII, both the pressure and temperature coefficients of longitudinal, shear, and bulk moduli evaluated at zero-pressure and 298°K are listed. The quantities of our interest $(\partial \ln M/\partial T)_V$, found from these values, are entered in the last column. Any assumption that an elastic modulus is a unique function

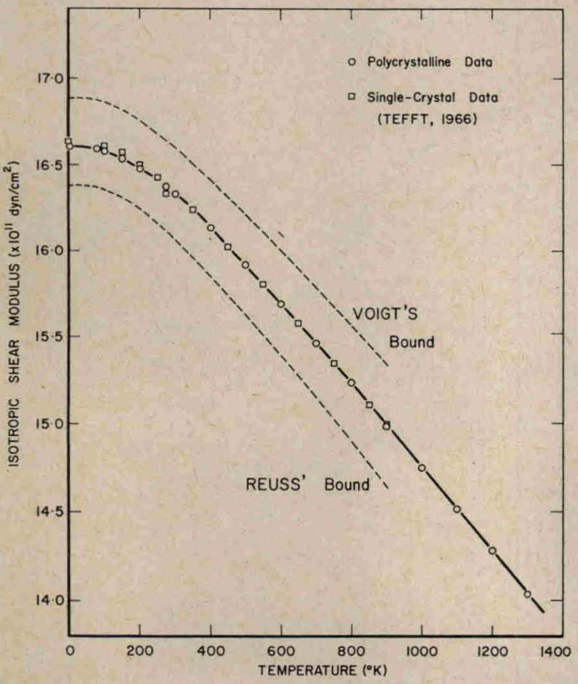

FIG. 4. A comparison between the measured and calculated isotropic shear modulus as a function of temperature.

TABLE VIII. Pressure and temperature coefficients of the adiabatic elastic moduli of polycrystalline α-Al_2O_3 (at 298°K).

Adiabatic elastic modulus	$d(\ln M)/dT$ ($\times 10^{-5}$/°K)	$\beta B^T[\partial(\ln M)/\partial p]_T$[a] ($\times 10^{-5}$/°K)	$[\partial(\ln M)/\partial T]_V$ ($\times 10^{-5}$/°K)
L^s	−8.23	+5.56	−2.67
G	−9.79	+4.45	−5.34
B^s	−6.664	+6.658	−0.006

[a] Based on Eq. (5.3), where $\beta=1.641\times 10^{-5}$/°K and $B^T=25.346\times 10^{11}$ dyn/cm².

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[25] D. Lazarus, Phys. Rev. **76**, 545 (1949).